

PLENTY OF MORSE FUNCTIONS BY PERTURBING WITH SUMS OF SQUARES

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ABSTRACT. We prove that given a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a submanifold $M \subset \mathbb{R}^n$, then the set of $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that $(f + q_a)|_M$ is Morse, where $q_a(x) = a_1x_1^2 + \dots + a_nx_n^2$, is a residual subset of \mathbb{R}^n . A standard transversality argument seems not to work and we need a more refined approach.

1. INTRODUCTION

The following problem was posed to the author by K. Kurdyka, to which we express our gratitude for stimulating discussions.

Given $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ we define the function $q_a : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$q_a(x) = a_1x_1^2 + \dots + a_nx_n^2$$

where in the case $b \in \mathbb{R}^k, k \neq n$, we mean q_b to belong to $C^\infty(\mathbb{R}^k, \mathbb{R})$ and to be defined in the same similar way.

Suppose $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $M \subset \mathbb{R}^n$ is a submanifold. Is it true that the set

$$A(f, M) = \{a \in \mathbb{R}^n \mid (f + q_a)|_M \text{ is Morse}\}$$

is residual in \mathbb{R}^n ?

The answer to this question turns out to be affirmative, but in a subtle way: standard transversality arguments based on dimension counting do not work and we have to prove it directly.

2. FAILURE OF PARAMETRIC TRANSVERSALITY ARGUMENT

We describe here what it is the usual procedure to prove that given a family of functions $f_a : M \rightarrow \mathbb{R}$ depending smoothly on the parameter $a \in A$ then the set of a such that f_a is Morse is residual in A .

Let $G : A \times M \rightarrow N$ be a smooth map and for every $a \in A$ let $g_a : M \rightarrow N$ be the function defined by $x \mapsto G(a, x)$. Suppose that $Z \subset N$ is a submanifold and that F is transverse to Z . Then from the *parametric transversality theorem* (see [2], Theorem 2.7) it follows that $\{a \in A \mid g_a \text{ is transverse to } Z\}$ is residual in A .

In the case we want to get Morse condition consider $N = T^*M$, $G(a, x) = d_x f_a$ and $Z \subset T^*M$ the zero section. Then f_a is Morse if and only if g_a is transverse to Z .

In our case, letting $M = \mathbb{R}^n$, we are led to define $G : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$(a, x) \mapsto (\partial f / \partial x_1(x) + a_1x_1, \dots, \partial f / \partial x_n(x) + a_nx_n)$$

and we consider $Z = \{0\} \in \mathbb{R}^n$. Then $f_a = f + q_a$ is Morse if and only if g_a is transversal to $\{0\}$. A condition that would ensure this (through the parametric

transversality theorem) is that G is transverse to $\{0\}$. Computing the differential of G at the point (a, x) we have for $(v, w) \in T_{(a, x)}(\mathbb{R}^n \times \mathbb{R}^n)$

$$(d_{(a, x)}G)(v, w) = \text{He}(f)(x)v + \text{diag}(a_1, \dots, a_n)v + \text{diag}(x_1, \dots, x_n)w$$

and we see that in general this condition does not hold (for example let $f \equiv 0$, then at all the points $(a, x) = (0, a_2, \dots, a_n, 0, \dots, 0)$ we have $G(a, x) = 0$ but $\text{rk}(d_{(a, x)}G) < n$).

3. A DIRECT APPROACH

First we recall the following Lemma (see [1]).

Lemma 1. *Let f be a smooth function on \mathbb{R}^n and for $a \in \mathbb{R}^n$ define the function f_a by $x \mapsto f(x) + a_1x_1 + \dots + a_nx_n$. The set*

$$\{a \in \mathbb{R}^n \mid f_a \text{ is Morse}\}$$

is residual in \mathbb{R}^n .

Proof. Define the function $g(x) = (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$ and notice that the Hessian of f is precisely the Jacobian of g and that x is a nondegenerate critical point for f if and only if $g(x) = 0$ and the Jacobian $J(g)(x)$ of g at x is nonsingular. Then $g_a(x) = g(x) + a$ and $J(g_a) = J(g)$. We have that x is a critical point for f_a if and only if $g(x) = -a$; moreover it is a nondegenerate critical point if and only if we also have $J(g)(x)$ is nonsingular, i.e. a is a regular value of g . The conclusion follows by Sard's lemma. \square

We immediately get the following corollary.

Corollary 2. *If f is a smooth function on an open subset U of \mathbb{R}^n such that for every $u = (u_1, \dots, u_n) \in U$ we have $u_i \neq 0$ for all $i = 1, \dots, n$, then*

$$A(f, U) = \{a \in \mathbb{R}^n \mid f + q_a \text{ is Morse on } U\}$$

is a residual subset of \mathbb{R}^n .

Proof. The functions u_1^2, \dots, u_n^2 are coordinates on U by hypothesis; we let \tilde{f} be the function f in these coordinates (it is defined on a certain open subset W of \mathbb{R}^n). Then for every $a \in \mathbb{R}^n$ we have that (using the above notation) \tilde{f}_a is Morse on W if and only if $f + q_a$ is Morse on U and the conclusion follows applying the previous lemma. \square

To prove the general statement we need the following.

Lemma 3. *Let f be a smooth function on an (arbitrary) open subset U of \mathbb{R}^n . Then the set $A(f, U)$ is residual in \mathbb{R}^n .*

Proof. For every $I = \{i_1, \dots, i_j\} \subset \{1, \dots, n\}$ define

$$H_I = U \cap \{u_i = 0, i \in I\} \cap \{u_k \neq 0, k \notin I\}.$$

To simplify notations let $I = \{1, \dots, j\}$. Notice that if $a = (a_1, \dots, a_n)$ and $a'' = (a_{j+1}, \dots, a_n)$ then $(q_a)|_{H_I} = (q_{a''})|_{H_I}$ where $q_{a''} : \mathbb{R}^{n-j} \rightarrow \mathbb{R}$ is defined as above. By corollary 2 the set

$$A''(f, H_I) = \{a'' \in \mathbb{R}^{n-j} \mid f|_{H_I} + q_{a''} \text{ is Morse on } H_I\}$$

is residual in \mathbb{R}^{n-j} . Let $a = (a', a'') \in \mathbb{R}^n$ such that $a'' \in A''(f, H_I)$ and suppose $x \in H_I$ is a critical point of $f + q_a$; then x is also a critical point of $(f + q_a)|_{H_I} =$

$f|_{H_I} + q_{a''}$. Since $a'' \in A''(f, H_I)$ then x belongs to a countable set, namely the set $C_{a''}$ of critical points of $f|_{H_I} + q_{a''}$ (each of this critical point must be nondegenerate by the choice of a''); moreover we have that

$$\text{He}(f|_{H_I} + q_{a''})(x) = \text{He}(f|_{H_I})(x) + \text{diag}(a_{j+1}, \dots, a_n)$$

is nondegenerate. Notice that the Hessian of $f + q_a$ at x is a block matrix:

$$\text{He}(f + q_a)(x) = \left(\begin{array}{c|c} \text{diag}(a_1, \dots, a_j) + B(x) & C(x) \\ \hline C(x)^T & \text{He}(f|_{H_I} + q_{a''})(x) \end{array} \right).$$

Thus for every $a'' = (a_{j+1}, \dots, a_n) \in A''(f, H_I)$ and for every $x \in C_{a''}$ consider the polynomial $p_{a'',x} \in \mathbb{R}[t_1, \dots, t_j]$ defined by

$$p_{a'',x}(t_1, \dots, t_j) = \det(\text{He}(f)(x) + \text{diag}(t_1, \dots, t_j, a_{j+1}, \dots, a_n))$$

Then the term of maximum degree of $p_{a'',x}$ is

$$t_1 \cdots t_j \det(\text{He}(f|_{H_I} + q_{a''})(x))$$

which is nonzero since $\det(\text{He}(f|_{H_I} + q_{a''})(x)) \neq 0$ (x is a *nondegenerate* critical point of $f|_{H_I} + q_{a''}$). It follows that $p_{a'',x}$ is not identically zero; hence its zero locus is a *proper* algebraic set. Thus for each $a'' \in A''(f, H_I)$ and each $x \in C_{a''}$ the set $A'(a'', x, I)$ defined by

$$\{a' \in \mathbb{R}^j \mid \text{if } x \text{ is a critical point of } f + q_{(a', a'')} \text{ on } H_I \text{ then it is nondegenerate}\}$$

is residual in \mathbb{R}^j (it is the complement of a proper algebraic set); it follows that

$$A'(a'', I) = \{a' \in \mathbb{R}^j \mid \text{each critical point of } f + q_{(a', a'')} \text{ on } H_I \text{ is nondegenerate}\}$$

is residual in \mathbb{R}^j , since it is a countable intersection of residual sets, i.e.

$$A'(a'', I) = \bigcap_{x \in C_{a''}} A'(a'', x, I)$$

Thus the set

$$A(f, I) = \{(a', a'') \mid a'' \in A''(f, H_I), a' \in A'(a'', I)\}$$

(which coincides with the set of $a = (a', a'') \in \mathbb{R}^n$ such that each critical point of $f + q_a$ on H_I is nondegenerate) is residual: is residual in a' for every a'' belonging to a residual set. Finally

$$A(f, U) = \bigcap_{I \subset \{1, \dots, n\}} A(f, I)$$

is a finite intersection of residual sets, hence residual. \square

Theorem 4. *Let f be a smooth function on \mathbb{R}^n and $M \subset \mathbb{R}^n$ be a submanifold. Then the set $A(f, M)$ is residual in \mathbb{R}^n .*

Proof. We basically improve the proof of Proposition 17.18 of [1].

Let $u_1, \dots, u_n : \mathbb{R}^n \rightarrow \mathbb{R}$ be the coordinates on \mathbb{R}^n . Suppose M is of dimension m . For every point $\bar{x} \in M$ there exists a neighborhood W of \bar{x} in M such that u_{i_1}, \dots, u_{i_m} are coordinates for M on

$$W \simeq \mathbb{R}^m,$$

for some $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$; since M is second countable, then it can be covered by a countable (finite if M is compact) number of such open sets. For convenience of notations suppose $\{i_1, \dots, i_m\} = \{1, \dots, m\}$.

Thus u_1, \dots, u_m are coordinates on $W \simeq \mathbb{R}^m$ and $f|_W, u_{m+1}|_W, \dots, u_n|_W$ are functions of $u_1|_W, \dots, u_m|_W$. Fix $a'' = (a_{m+1}, \dots, a_n) \in \mathbb{R}^{n-m}$ and define $g_{a''} : W \rightarrow \mathbb{R}$ by

$$g_{a''} = f|_W + a_{m+1}u_{m+1}^2|_W + \dots + a_nu_n^2|_W = (f + a_{m+1}u_{m+1}^2 + \dots + a_nu_n^2)|_W$$

Notice that $g_{a''}$ is not $(f + q_a)|_W$ since we are taking only the last $n - m$ of the a'_i 's; we still have the freedom of choice (a_1, \dots, a_m) .

By lemma 3, since $u_1|_W, \dots, u_m|_W$ are coordinates on W , for every $a'' \in \mathbb{R}^{n-m}$ the set

$$\{a' = (a_1, \dots, a_m) \in \mathbb{R}^m \text{ s.t. } g_{a''} + a_1u_1^2|_W + \dots + a_mu_m^2|_W \text{ is Morse on } W\}$$

is residual in \mathbb{R}^m . Notice that $g_{a''} + a_1u_1^2|_W + \dots + a_mu_m^2|_W = (f + q_{(a', a'')})|_W$; hence for every a'' the set of a' such that $(f + q_{(a', a'')})|_W$ is Morse on W is residual. Thus the set of $a \in \mathbb{R}^n$ such that $(f + q_a)|_W$ is Morse on W is residual (it is residual in a' for each fixed a'' hence it is globally residual). It follows that $A(f, M)$ is a countable intersection of residual set, hence residual. \square

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